Phugoid Oscillations at Hypersonic Speeds

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A theoretical analysis is made of the long-period (phugoid) oscillations of a lifting vehicle in hypersonic flight up to and including orbital speeds in any atmosphere. These oscillations correspond to a direct exchange between the kinetic energy and the potential energy along an oscillating flight path governed primarily by the trimmed steady-state zero pitching moment. The expressions derived for the period and damping of these oscillations show that for all speeds near or greater than sonic the atmospheric density gradient has an important effect in decreasing the period and increasing the damping of the phugoid oscillations. As orbital speeds are approached, it is found that the decrease in gravitational attraction with altitude overcomes the effect of the decrease in atmospheric density with altitude so that with increasing speed the phugoid period asymptotically approaches the corresponding satellite orbit period. In addition, simple explicit expressions are derived for the so-called "short period" or angle-of-attack oscillations. These and the phugoid relations are shown to be in excellent agreement with the numerical calculations previously presented by Etkin.

Nomenclature

a	= speed of sound
B	= pitching moment of inertia about the y axis
$F = U/(gR)^{1/2}$	= ratio of trajectory velocity to circular orbit velocity
g	= gravitational acceleration
L, S	= reference length and reference area
$L_0 = \frac{1}{2}\rho U^2 S C_L$	= steady-state lift force $= mg(1 - F^2)$
M	= pitching moment or Mach number $M = U/a$
q	= pitching velocity about vehicle's c.g.
r	= radius measured from center of the earth to the vehicle's c.g. (see Fig. 2)
R	= radius measured from the center of the earth to the steady-state constant altitude trajectory
U	= steady-state or average velocity of flight
u, w	= velocity components along and perpendicular to fixed body axes
X, Z	= forces along fixed body axes
$\alpha = \tan^{-1}(w/u)$	= angle of attack
γ	= flight path angle relative to the earth's surface
θ	= angle of principal body x axis relative to the earth's surface (see Figs. 2 and 3)
ρ	= mass density of atmosphere
$\rho' = \partial\rho/\partial r$	= atmospheric density gradient
$\mu = \rho SL/2m \ll 1$	= nondimensional ratio proportional to mass of displaced air to actual mass (m) of vehicle
$\sigma = mL^2/B$	= nondimensional reciprocal of the square of the radius of gyration
C_L, C_D	= coefficients for lift and drag forces perpendicular and parallel to the flight path, nondimensionalized by dividing by $\frac{1}{2}\rho U^2 S$
C_m	= pitching moment coefficient, nondimensionalized by dividing by $\frac{1}{2}\rho U^2 SL$

1. Introduction

EXPLICIT expressions are derived for both the long-period (phugoid) oscillations and the short-period (angle-of-attack) oscillations of a hypersonic nonrolling missile having longitudinal oscillations in its plane of symmetry. The phugoid oscillations correspond to the classical investigation of Lanchester[1] for the relatively slow pitching motion at nearly constant angle of attack and nearly zero pitching moment. In this long period oscillation the propulsive thrust is assumed to nearly balance the drag force so that the phugoid oscillation corresponds to a very lightly damped exchange between the kinetic energy and the potential energy at a fixed constant value of the lift coefficient. On the other hand, the short-period oscillation usually consists of a relatively rapid, very highly damped variation in the angle of attack that is over so quickly that the flight velocity (or kinetic energy) remains nearly constant, whereas the altitude (or potential energy) has only been changed by a negligible amount.

Etkin[2] was the first to show by direct numerical calculations that at hypersonic speeds the so-called, short-period oscillations can develop a period that is in reality longer than that of the corresponding phugoid if the flight altitude is sufficiently high so that the aerodynamic restoring moment becomes relatively small. Consequently, we will refer to the so-called, short-period oscillations as the angle-of-attack oscillations, as opposed to the phugoid oscillations that occur at a nearly constant value of the angle of attack.

Lanchester[1] assumed exactly zero net drag, a constant lift coefficient, a negligible moment of inertia about the pitching axis, and a constant atmospheric density in order to derive an undamped phugoid period of

$$T = (2)^{1/2}(U/g)\pi \tag{1.1}$$

for any lifting aircraft trimmed to fly with zero pitching moment at a constant altitude or horizontal velocity $= U$, and having negligible damping forces or moments.

Scheubel[3] was the first to show that the actual variation of the air density with altitude would have the effect of decreasing the phugoid period, especially as the speed increased. By using the same procedure and assumptions that Lanchester[1] introduced, Scheubel[3] derived the undamped phugoid period in a stratified atmosphere as

$$T = (2)^{1/2}\frac{U}{g}\pi\left[1 + \frac{U^2}{2g}\left(-\frac{\rho'}{\rho}\right)\right]^{-1/2} \tag{1.2}$$

Since, in the earth's atmosphere, we have

$$\frac{\rho'}{\rho} = \frac{1}{\rho}\frac{\partial\rho}{\partial r} \approx -(22 \times 10^3 \text{ ft})^{-1} \tag{1.3}$$

therefore, the phugoid period would be decreased by approximately 30% at sonic speeds, and even more drastically shortened by increasing supersonic speeds.

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Neumark[4] then noted that compressibility effects must now be included, so he analyzed the linearized equations of motion for disturbed horizontal flight over a flat earth. He derived a fifth-order equation for the eigenvalues of the combined equations of motion, with compressibility effects included in the aerodynamic forces and moments. These equations and the approximate roots that Neumark[4] derived describe all the necessary characteristics of various supersonic aircraft that cannot attain hypersonic speeds. For example, he showed that an approximate factorization for the damped long-period oscillations could be reduced at supersonic speeds to

$$T = \frac{(2)^{1/2}\, U/g\pi}{\{1 + (U^2/2g)[-(\rho'/\rho)] + [1 - (U^2/2g)[-(a'/a)\zeta]]\}^{1/2}}$$

$$\frac{a'}{a} = \frac{1}{a}\frac{\partial a}{\partial r} = -\frac{1}{2}\left(\frac{\rho'}{\rho} + \frac{\rho g}{p}\right) \tag{1.4}$$

$$\zeta = \frac{M}{2C_L}\left(\frac{\partial C_L}{\partial M}\right)_\alpha \approx -\frac{1}{2}\frac{M^2}{M^2 - 1}$$

Etkin[2] then showed by numerical calculation that at hypersonic speeds one must also include the centrifugal force effects produced by the curvature of the steady flight path about a spherical earth. In addition, as orbital speeds were approached, the variation of the earth's gravity force with altitude also became important. Consequently, the purpose of the present paper is to derive an explicit expression exhibiting all of these effects for hypersonic speeds, wherein all of the aerodynamic coefficients become independent of the Mach number. Then we find that the phugoid period, at a small constant angle of attack and with a zero net drag force, is given by

$$T = \frac{(2)^{1/2}\,(U/g)\pi}{\left\{1 + \frac{U^2}{2g}\left[-\frac{\rho'}{\rho}(1 - F^2) - \frac{(2 - F^2)}{R}\right]\right\}^{1/2}} \tag{1.5}$$

$$F^2 = U^2/gR \approx (\text{fps}/26{,}400)^2 < 1 \qquad R > 21 \times 10^6\ \text{ft}$$

where R is the radius of the steady flight path, measured from the center of the spherical earth. Figure 1 shows that the effect of the flight path curvature, which is included in Eq. (1.5), cannot be neglected when $F > 0.4$ or $U > 10{,}600$ fps.

2. Period for the Phugoid with No Drag Force and No Damping

If we assume a constant life coefficient and a zero drag force (i.e., the thrust exactly cancels all of the drag forces)

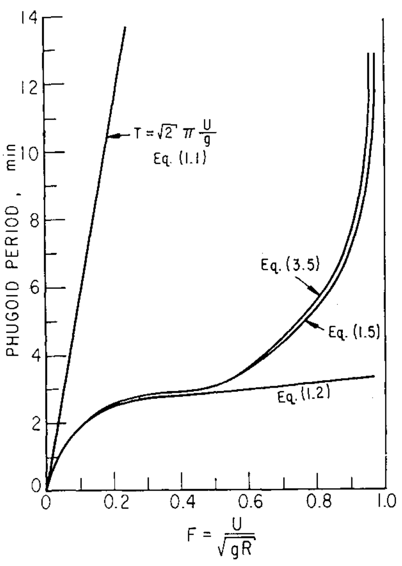

Fig. 1 Phugoid period.

with zero pitching acceleration, then it is advantageous to use the axes system that is always tangent to the flight path as shown in Fig. 2 so that

$$m(dV/dt) = -mg(R/r)^2 \sin\gamma \approx -mg(R/r)^2\gamma$$

$$mV(d\gamma/dt) = \tfrac{1}{2}\rho V^2 SC_L - m[g(R/r)^2 - (V^2/r)]\cos\gamma \tag{2.1}$$

$$B[d^2(\gamma + \alpha - \Phi)/dt^2] = \tfrac{1}{2}\rho V^2 SLC_m \approx 0$$

Then we can linearize Eq. (2.1) for the slow oscillations corresponding to the phugoid mode for a flight path that is nearly parallel to the earth's surface by writing

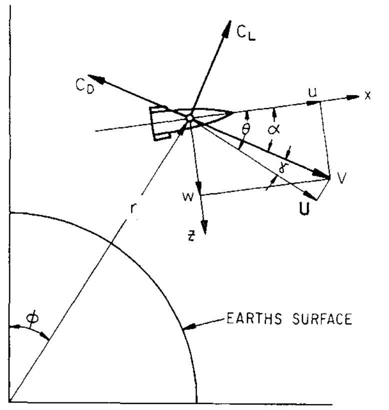

Fig. 2 Axes system tangent to flight path.

$$\frac{1}{V}\frac{dr}{dt} = \sin\gamma \approx \gamma \qquad \frac{d\gamma}{dt} \approx \frac{1}{V}\frac{d^2r}{dt^2} - \frac{1}{V^2}\frac{dr}{dt}\frac{dV}{dt}$$

$$r(t) = R\{[1 + \epsilon(t)]\} \qquad V(t) = U[1 + \eta(t)]$$

$$\rho(r) = \rho(R)[1 + (\rho'/\rho)\epsilon(t)R]$$

$$U\frac{d\eta}{dt} \approx -\frac{gR}{U}\frac{d\epsilon}{dt} \qquad \eta \approx -\frac{gR}{U^2}\epsilon + 0(\epsilon^2)$$

$$\frac{d^2\epsilon}{dt^2} + \left[\frac{\rho(R)U^2SC_L}{2mR}\left(-\frac{\rho'}{\rho(R)}R + \frac{2gR}{U^2}\right) + \frac{U^2}{R^2}\right]\epsilon = 0(\epsilon^2) \tag{2.2}$$

Therefore, the period of the phugoid oscillation is given by

$$T = \frac{2\pi R}{U}\left\{1 + \frac{L_0}{mg}\left(\frac{gR}{U^2}\right)\left(-\frac{\rho'}{\rho}R + \frac{2gR}{U^2}\right)\right\}^{-1/2} \tag{2.3}$$

which may be written as Eq. (1.5) since

$$\frac{\rho U^2 SC_L}{2mg} = \frac{L_0}{mg} = \left(1 - \frac{U^2}{gR}\right) = (1 - F^2) \tag{2.4}$$

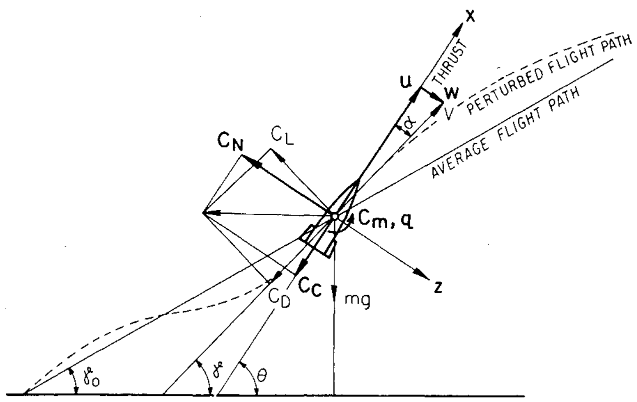

Fig. 3 Fixed body axis system.

It should be noted from Eq. (2.3) and Fig. 1 how the phugoid period asymptotically approaches the circular orbit period as $F \rightarrow 1$.

3. Period and Damping for the Phugoid Oscillations at Hypersonic Speeds

At hypersonic speeds, all of the aerodynamic coefficients are independent of the Mach number. Consequently, for the relatively slow phugoid oscillation, both the angle of attack and lift coefficient may be assumed to be small and constant, whereas the angular acceleration is negligible. In this case, Etkin's[2] equations for a spherical earth reduce to

$$X_u\Delta u + X_r\Delta r - mg\Delta\theta = m(d\Delta u/dt)$$

$$Z_u\Delta u + Z_r\Delta r - 2mg\frac{\Delta r}{R} = \left[-mq_0\Delta u - mu\left(\frac{d\theta}{dt} - \frac{\Delta u}{R} - q_0\frac{\Delta r}{R}\right)\right] \quad (3.1)$$

$$q = \left(\frac{d\theta}{dt} - \frac{\Delta u}{R} - q_0\frac{\Delta r}{R}\right)$$

$$\frac{dr}{dt} = U\Delta\theta \qquad g(r) = g(R)\left(\frac{R^2}{r^2}\right)$$

for axes that are fixed in the body as shown in Fig. 3.

The matrix form of Eq. (3.1) may be written as

$$\begin{bmatrix} \left(m\frac{d}{dt} - X_u\right) & (mg) & (-X_r) \\ \left(-Z_u - mq_0 + \frac{mU}{R}\right) & \left(-mU\frac{d}{dt}\right) & \left(-Z_r + \frac{2mg}{R} + \frac{mU}{R}q_0\right) \\ 0 & (-U) & \left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \\ \Delta r \end{bmatrix} = 0 \quad (3.2)$$

If we note from Table 1 that $X_rZ_u = X_uZ_r$ and $q_0 = -U/R$, then the roots (λ) of the characteristic equation of this matrix are given by

$$\left\{\lambda^3 - \frac{X_u}{m}\lambda^2 + \left(\frac{Z_r}{m} - \frac{g}{U}\frac{Z_u}{m} + \frac{U^2}{R^2}\right)\lambda + \frac{X_u}{m}\left(\frac{2g}{R} - \frac{U^2}{R^2}\right) + \frac{X_r}{m}\left(-\frac{2U}{R}\right)\right\} = 0 \quad (3.3)$$

where the last terms not containing λ represent the variation in the drag force. Because a change in the drag force cannot effect the phugoid period until after many cycles (e.g., see Refs. 2 and 4), we can therefore obtain the period from Eq. (3.3) as the imaginary part of λ_∞ in the reduced equation:

$$\lambda_\infty{}^2 - \frac{X_u}{m}\lambda_\infty + \left(\frac{Z_r}{m} - \frac{g}{U}\frac{Z_u}{m} + \frac{U^2}{R^2}\right) = 0 \quad (3.4)$$

so that (see Table 1)

$$T = (2)^{1/2}\frac{U}{g}\pi\left\{1 + \frac{U^2}{2g} \times \left[-\frac{\rho'}{\rho}(1 - F^2) - \frac{(2 - F^2)}{R} - \frac{1}{2}\left(\frac{D}{mg}\right)^2\right]\right\}^{-1/2}$$

$$F^2 = U^2/gR \qquad D = \tfrac{1}{2}\rho U^2 SC_D \quad (3.5)$$

Figure 1 shows the changes in the phugoid period due to all of the combined effects in Eq. (3.5), whereas Fig. 4 shows how well Eq. (3.5) agrees with Etkin's numerical calculations for the phugoid period.

If the thrust exactly cancels the drag force, then $C_D = 0$ and Eq. (3.5) reduces to Eq. (1.5). However, there is no damping of the phugoid oscillation unless $C_D > 0$. With a finite drag force, we may evaluate the real part of λ in Eq. (3.3) to determine the damping coefficient. Since the damping of the phugoid is small, as shown by the numerical calculations of Etkin,[2] we may assume that

$$\lambda = \lambda_\infty + \delta \qquad \delta \ll |\lambda| \quad (3.6)$$

where δ is a small real number and λ_∞ is the complex root of Eq. (3.4). Then upon substituting Eq. (3.6) into Eq. (3.3), neglecting terms of order δ^2, and retaining only the predominant terms, we obtain the first-order approximation for $C_D{}^2 \ll C_L$ as

$$\delta = \frac{(X_u/m)[(2g/R) - (U^2/R^2)] + (X_r/m)[-(2U/R)]}{2\{(U^2/R^2) + (\rho U^2SC_L/2m)[-(\rho'/\rho) + (2g/U^2)]\}} \quad (3.7)$$

Then the real part of λ gives the damping term as

$$\exp\left\{-\frac{\rho USC_D}{2m} - \left[\frac{-(\rho'/\rho)U + (2g/U) - (U/R)}{-(\rho'/\rho)R + (2/F^2) + (2m/\rho SRC_L)}\right]\frac{C_D}{C_L}\right\}t \quad (3.8)$$

or from Eq. (2.4) as

$$\exp\left\{-\frac{g}{U}\frac{D}{mg} - \left[\frac{-(\rho'/\rho)U + (2g/U) - (U/R)}{-(\rho'/\rho)R + (2/F^2) + (mgF^2/L_0)}\right]\frac{D}{L_0}\right\}t \quad (3.9)$$

Figure 5 shows the excellent agreement between Eq. (3.9) and Etkin's[2] numerical results for the damping of the phugoid oscillations with a constant thrust at hypersonic speeds. Equation (3.9) clearly proves that the damping due to the atmospheric density gradient becomes predominant at higher speeds and higher altitudes.

4. Angle of Attack Oscillations

The relatively rapid oscillations in angle of attack with $V \approx U = \text{const}$, which usually have a highly damped short period at altitudes below 200,000 ft (e.g., see Ref. 2), may be expressed for hypersonic speeds by means of the following equations (e.g., see Ref. 5) for axes fixed in the body along the principal axes of inertia (Fig. 3):

$$(d^2\alpha/dt^2) + b(d\alpha/dt) + c\alpha = 0 \quad (4.1)$$

$$b = \mu(U/L)(C_{N\alpha} - \sigma C_{mq}) + (1/U)(dU/dt) \quad (4.2)$$

$$c = \left\{\mu\left(\frac{U}{L}\right)^2\sigma(-C_{m\alpha} - \mu C_{mq}C_{N\alpha}) + \frac{U}{L}C_{N\alpha}\frac{d\mu}{dt} + \frac{\mu}{L}(C_{N\alpha} - \sigma C_{mq})\frac{dU}{dt} - \left(\frac{1}{U}\frac{dU}{dt}\right)^2 + \frac{d^2U}{dt^2}\frac{1}{U}\right\} \quad (4.3)$$

where

$$\mu = \frac{\rho SL}{2m} < 10^{-3} \qquad \sigma = \frac{mL^2}{B} \geqslant 6$$

$$C_{N\alpha} = (C_{L\alpha} + C_D) \qquad C_{mq} = \frac{\partial C_m}{\partial(Lq/U)} \quad (4.4)$$

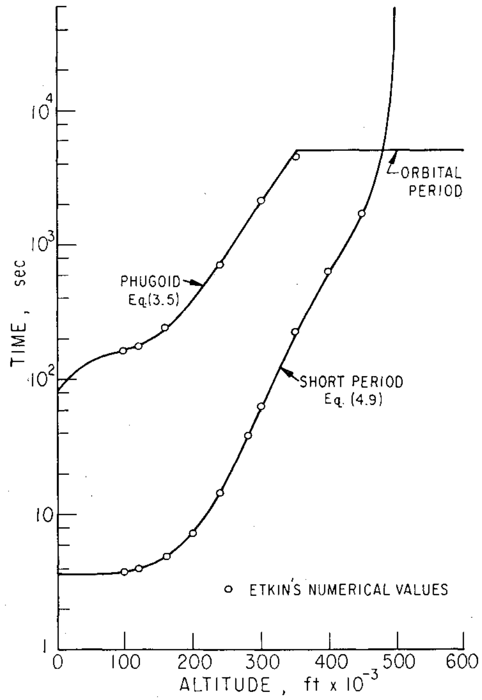

Fig. 4 Period.

whereas the rest of the symbols are defined in Table 1. The acceleration terms are found to be important at hypersonic speeds only for the re-entry problem involving high rates of deceleration. Consequently, for the steady hypersonic flight path parallel to the earth's surface ($\gamma_0 = 0$) we can reduce Eqs. (4.2) and (4.3) to

$$b \approx \mu(U/L)(C_{L\alpha} + C_D - \sigma C_{mq}) \tag{4.5}$$

$$c \approx \mu(U/L)^2 \; (-\; C_{m\alpha})\sigma \tag{4.6}$$

since

$$-\; C_{m\alpha} \gg |\mu C_{mq}(C_{L\alpha} + C_D)|$$

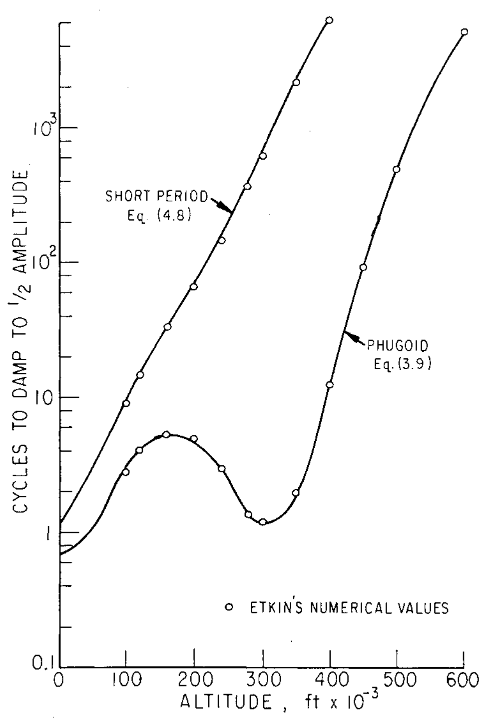

Fig. 5 Damping.

Table 1 Aerodynamic stability derivatives for longitudinal oscillations at hypersonic speeds with constant thrust when $\alpha \approx 0$

$-X = \frac{1}{2}\rho V^2 S C_C = \frac{1}{2}\rho V^2 S(C_D \cos\alpha - C_L \sin\alpha)$	
$-X_u = \rho U S C_D \left(1 + \frac{U}{2}\frac{\rho'}{\rho}\frac{dr}{du}\right) \approx \rho U S C_D$	
$-X_w = \frac{1}{2}\rho U S(C_{D\alpha} - C_L)$	
$-X_q = \frac{1}{2}\rho U S L \frac{\partial C_D}{\partial(Lq/U)}$	
$-X_r = \frac{1}{2}\rho U^2 S C_D(\rho'/\rho)$	
$-Z = \frac{1}{2}\rho V^2 S C_N = \frac{1}{2}\rho V^2 S(C_L \cos\alpha + C_D \sin\alpha)$	
$-Z_u = \rho U S C_L \left(1 + \frac{U}{2}\frac{\rho'}{\rho}\frac{dr}{du}\right) \approx \rho U S C_L$	
$-Z_w = \frac{1}{2}\rho U S(C_{L\alpha} + C_D)$	
$-Z_q = \frac{1}{2}\rho U S L \frac{\partial C_L}{\partial(Lq/U)}$	
$-Z_r = \frac{1}{2}\rho U^2 S C_L(\rho'/\rho)$	
$M = \frac{1}{2}\rho V^2 S L C_m$	
$M_u = 0 = M_r$	$M_{\dot{w}} \approx 0 \approx C_{m\dot{\alpha}}$
$M_w = \frac{1}{2}\rho U S L \frac{\partial C_m}{\partial\alpha}$	$M_q = \frac{1}{2}\rho U S L^2 \frac{\partial C_m}{\partial(Lq/U)}$

Therefore, for hypersonic speeds we can write for the period

$$T \approx 2\pi(L/U)(-\;\mu\sigma C_{m\alpha})^{-1/2} \tag{4.7}$$

whereas for the damping we obtain

$$\exp\{-(\mu/2)(U/L)(C_{L\alpha} + C_D - \sigma C_{mq})\}t \tag{4.8}$$

The comparison with Etkin's[2] numerical calculations in Fig. 5 shows that Eq. (4.8) gives a satisfactory approximation to the damping of the angle-of-attack oscillations at all hypersonic speeds, whereas a comparison of Eq. (4.7) for the period shows that it is adequate for all altitudes below 400,000 ft. However, at higher altitudes the aerodynamic forces and moments practically vanish so we must add the extremely small gravity gradient effect upon the mass distribution of the body itself. This was given by Etkin[2] for his numerical example as

$$\alpha\frac{\partial C_m}{\partial\theta} \approx \frac{1.41 mgL}{\frac{1}{2}\rho U^2 SR} = \frac{1.41}{\mu}\left(\frac{gL^2}{RU^2}\right) \geqslant \frac{1.41}{\mu}\left(\frac{L}{R}\right)^2$$

Actually, for any conventional body we have

$$\left|\frac{\partial C_m}{\partial\theta}\right| < \frac{3(U/R)^2 B}{\frac{1}{2}\rho U^2 SL} = \frac{3}{\mu\sigma}\left(\frac{L}{R}\right)^2$$

so that Eq. (4.7) could then be replaced by

$$T \approx 2\pi(L/U)[\mu\sigma(-C_{m\alpha}) - 3(L/R)^2]^{-1/2} \tag{4.9}$$

As shown in Fig. 4, this Eq. (4.9) can be seen as a special case of Etkin's[2] Eq. (4.1) and is in excellent agreement with his numerical values at all altitudes.

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